Fluctuations and moderate deviations for a catalytic Fleming-Viot branching system in nonequilibrium

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Outline

Background and Model

Fluctuations

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The catalytic Fleming-Viot branching system

- The catalytic Fleming-Viot branching system is a jump diffusion process describing a system of diffusing particles (see Grigorescu [3]).
- The hydrodynamic limit for the empirical measure is the solution to a generalized semilinear (reaction-diffusion) equation, with nonlinearity given by a quadratic operator.

- d-dimensional unit torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.
- $V(x)$ is a bounded continuous function on $\mathbb{T}^d$.
- For $\boldsymbol{x} = (x_1, x_2, \cdots, x_N) \in (\mathbb{T}^d)^N$, $\boldsymbol{x}^{ij} \in (\mathbb{T}^d)^N$ is the vector where the component i has been deleted and replaced with the component j for all $1 \le i \ne j \le N$.
- $h \in C^{1,2}([0,\infty) \times \mathbb{T}^d)$, $H(t, \boldsymbol{x}) = \sum_{i=1}^N h(t, x_i)$, and

$$p_{ij}^{N,h}(t, \boldsymbol{x}) = p_{ij} = \frac{1}{N-1} e^{H(t,\boldsymbol{x}^{ij}) - H(t,\boldsymbol{x})}, \ j \ne i, \quad p_{ii}^{N,h} = p_{ii} = 0. \tag{1.1}$$

- $\zeta(dx)$ is a probability measure on $(\mathbb{T}^d)^N$.
- $P^N_{\zeta,H}$ is a probability measure on $D([0,\infty),(\mathbb{T}^d)^N)$ such that under $P^N_{\zeta,H}$, the coordinate process

$$\{\boldsymbol{x}(t) = (x_1(t), x_2(t), \cdots, x_N(t)), t \geq 0\}$$

is a Feller process with the generator $\mathcal{L}^{N,h}_t$ defined by

$$\mathcal{L}^{N,h}_t f(t,\boldsymbol{x}) = \sum_{i=1}^{N} \left(\frac{1}{2} \triangle_{x_i} f(t,\boldsymbol{x}) + \nabla_{x_i} H(t,\boldsymbol{x}) \cdot \nabla_{x_i} f(t,\boldsymbol{x}) \right) + \sum_{i=1}^{N} \int_0^t \sum_{j \neq i} p^{N,h}_{ij}(t,\boldsymbol{x})(f(t,\boldsymbol{x}^{ij}) - f(t,\boldsymbol{x}))V(x_i), \tag{1.2}$$

for $f \in C^{1,2}([0,\infty) \times (\mathbb{T}^d)^N)$, where $x \cdot y$ denotes the inner product.

- The process $\{\boldsymbol{x}(t), P^N_{\zeta,h}\}$ exists, and it is the solution of the following martingale problem: for any $f \in C^{1,2}([0,\infty) \times (\mathbb{T}^d)^N)$,

$$\begin{aligned} M_t^{N,h,f} =& f(t,\boldsymbol{x}(t)) - f(0,\boldsymbol{x}(0)) \\ & - \int_0^t \Big(\partial_s f(s,\boldsymbol{x}(s)) + \mathcal{L}_t^{N,h} f(s,\boldsymbol{x}(s)) \Big) ds \end{aligned} \tag{1.3}$$

is a P-martingale with

$$\begin{aligned} \langle M^{N,h,f} \rangle_t =& \frac{1}{2} \int_0^t \Big(\sum_{i=1}^N |\nabla_{x_i} f(s,\boldsymbol{x}(s))|^2 + \sum_{j \neq i} p_{ij}^{N,h}(s,\boldsymbol{x}(s)) \\ & \times (f(s,\boldsymbol{x}^{ij}(s)) - f(s,\boldsymbol{x}(s))) V(x_i(s)) \Big) ds. \end{aligned}$$

- The process $\{\{\boldsymbol{x}(t), t \geq 0\}, P^N_{\zeta,h}\}$ is called a catalytic Fleming-Viot branching system.

- The catalytic Fleming-Viot branching system with uniform redistribution mechanism, i.e., $h = 0$.
- P_γ^N denotes the law of the process starting at

$$\zeta(d\boldsymbol{x}) = \otimes_{j=1}^N \gamma(dx_j), \quad \gamma(dx) = \gamma(x)dx,$$

  with bounded initial density $\gamma(x)$.
- The expectation with respect to P_γ^N is denoted by E_γ^N.
- The empirical measure process

$$\mu_t^N(d\boldsymbol{x}) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)} \in D([0,T], M_1(\mathbb{T}^d)), \;\; 0 \le t \le T. \tag{1.4}$$

- Consider a linear operator on the space $D([0,\infty), M_b(\mathbb{T}^d))$,

$$a : \mu \to a(\mu)(t,x) = \langle \mu_t, V\rangle - V(x). \tag{1.5}$$

- A measure-valued path $\{\rho_t(dx), t \geq 0\}$ is the unique weak solution of the integro-differential equation

$$\partial_t \mu = \frac{1}{2} \triangle \mu + \mu a(\mu), \quad \mu_0 = \gamma. \tag{1.6}$$

- The hydrodynamic limit (Grigorescu [3]), i.e., for any $\phi \in C^{1,2}([0, T] \times \mathbb{T}^d)$, any $\varepsilon > 0$,

$$\lim_{N \to \infty} P_\gamma^N \left(\sup_{t \in [0,T]} |\langle \mu_t^N(\cdot) - \rho_t(\cdot), \phi(t, \cdot) \rangle| \geq \varepsilon \right) = 0. \tag{1.7}$$

- The large deviations for the empirical measure process (Grigorescu [3]).

Our purpose

- Fluctuation: The weak convergence of the empirical fluctuation fields $\eta_t^N(dx)$, $N \geq 1$ defined by

$$\eta_t^N(dx) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\delta_{x_i(t)}(dx) - \rho_t(dx)), \qquad (1.8)$$

- The moderate deviation principle: The large deviation principle of the centralized empirical measure process

$$\widetilde{\eta}_t^N(dx) = \frac{1}{a(N)} \sum_{i=1}^{N} (\delta_{x_i(t)}(dx) - \rho_t(dx)) = \frac{\sqrt{N}}{a(N)} \eta_t^N(dx), \qquad (1.9)$$

where $\{a(t), t \geq 0\}$ is a positive function with

$$\lim_{t\to\infty} a(t)/\sqrt{t} = \infty, \qquad \lim_{t\to\infty} a(t)/t = 0, \qquad (1.10)$$

Fluctuations

- For every integer m, for each $g \in C^\infty(\mathbb{T}^d)$, define

$$\|g\|_m = \left(\sum_{|k| \le m} \int_{\mathbb{T}^d} |\partial^k g(x)|^2 dx \right)^{1/2} < \infty.$$

  Let $\mathbb{H}^m$ be the complete of $(C^\infty(\mathbb{T}^d), \|\cdot\|_m)$, and $\mathbb{H}^{-m}$ the dual space of $\mathbb{H}^m$.
- Let $\frac{1}{2}\triangle$ be the Laplace operator on $\mathbb{T}^d$ and let $U(t)$ be the heat semigroup associated with $\frac{1}{2}\triangle$ on $\mathbb{T}^d$.

- **(A0).** $h(t,x) \equiv 0$, and V is a non-negative continuous function on $\mathbb{T}^d$ with partial derivatives up to order $(5+3D)$, where $D = [d/2] + 1$.
- Let the condition **(A0)** hold. For $m \geq 1 + D$, let W be the continuous Gaussian martingale process taking its values in $\mathbb{H}^{-m}$ with mean 0 and variance given by

$$
\begin{aligned}
&E\left(W_t(\varphi)^2\right) \\
&= \int_0^t \left(\frac{1}{2}\langle \rho_s, |\nabla\varphi|^2\rangle + \langle \rho_s, \varphi^2\rangle\langle \rho_s, V\rangle - 2\langle \rho_s, \varphi\rangle\langle \rho_s, \varphi V\rangle + \langle \rho_s, \varphi^2 V\rangle \right) ds
\end{aligned} \tag{2.1}
$$

  for every $\varphi \in \mathbb{H}^m$ and $t \in [0, T]$.
- Let F_t be the operator on $\mathbb{H}^m$ defined by

$$
F_t\varphi(x) = \langle \rho_t, V\rangle\varphi(x) + \langle \rho_t, \varphi\rangle V(x) - V(x)\varphi(x). \tag{2.2}
$$

Fluctuation Theorem

Theorem 2.1 (Fluctuation Theorem)

Assume that the condition **(A0)** *holds. Then under* P_γ^N*, the sequence* $\{\eta^N, N \geq 1\}$ *converges in law to the generalized Ornstein-Uhlenbeck process* η *with catalyst* V *in* $\mathbb{D}\left([0, T], \mathbb{H}^{-(4+2D)}\right)$*. i.e., for any* $\varphi \in \mathbb{H}^{4+2D}$*,*

$$\langle \eta_t, \varphi \rangle = \langle \eta_0, U(t)\varphi \rangle + \int_0^t \langle \eta_s, F_s U(t-s)\varphi \rangle ds + \int_0^t \langle U(t-s)\varphi, dW_s \rangle, \tag{2.3}$$

- For any $t \in [0, T]$ and $\varphi \in C^2$, applying (1.3) to $f(t, \boldsymbol{x}(t)) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varphi(x_i(t))$, we have

$$\langle \eta_t^N, \varphi \rangle = \langle \eta_0^N, \varphi \rangle + \int_0^t \langle \eta_s^N, \frac{1}{2} \triangle \varphi \rangle ds + \int_0^t \langle \eta_s^N, F_s^N \varphi \rangle ds + M_t^N(\varphi), \tag{2.4}$$

- where

$$F_s^N \varphi(x) = \frac{N}{N-1} \langle \mu_s^N, V \rangle \varphi(x) + \langle \rho_s, \varphi \rangle V(x) - V(x)\varphi(x), \tag{2.5}$$

- $M_t^N(\varphi)$ is a square integrable martingale with

$$\langle M^N(\varphi) \rangle_t = \int_0^t \langle \mu_s^N, (\nabla \varphi)^2 \rangle ds + \frac{N}{N-1} \int_0^t \Big(\langle \mu_s^N, \varphi^2 \rangle \langle \mu_s^N, V \rangle - 2 \langle \mu_s^N, \varphi \rangle \langle \mu_s^N, \varphi V \rangle + \langle \mu_s^N, \varphi^2 V \rangle \Big) ds. \tag{2.6}$$

- Informally,
  - $M^N(\varphi) \to W(\varphi)$ follows from $\mu^N \to \rho$,
  - and so if η^N has a limit point η, then η satisfies (2.3).
- In order to give a rigorous proof, we need some moment estimates.
  - The sequence

$$\left\{\left\{(M_t^N, \eta_t^N), t \in [0, T]\right\}, N \geq 1\right\}$$

    is tight in $D([0, T], \mathbb{H}^{-(4+2D)} \times \mathbb{H}^{-(4+2D)})$.
  - All limit points of the sequence $\{\mathcal{L}((M^N, \eta^N)), N \geq 1\}$ charge only in $C([0, T], \mathbb{H}^{-(4+2D)} \times \mathbb{H}^{-(4+2D)})$.
  - Let (M, η) be a weak limit point of the sequence $\{(M^N, \eta^N), N \geq 1\}$ in $D([0, T], \mathbb{H}^{-(4+2D)} \times \mathbb{H}^{-(4+2D)})$. Then M has the same law as W, and (W, η) solves the equation (2.3).

Moderate deviations

Theorem 3.1 (Moderate deviations)

Assume that the condition **(A0)** *holds. Then the sequence* $\{\widetilde{\eta}^N, N \geq 1\}$ *satisfies a large deviation principle on* $D([0,T], \mathbb{H}^{-(5+3D)})$ *with the speed* $a^2(N)/N$ *and the good rate function* I *defined by*

$$
\begin{aligned}
I(\nu) = & \sup_{\psi \in C(\mathbb{T}^d)} \left\{ \langle \nu_0, \psi \rangle - \frac{1}{2} \left(\int_{\mathbb{T}^d} |\psi(x)|^2 \gamma(x) dx - \left(\int_{\mathbb{T}^d} \psi(x) \gamma(x) dx \right)^2 \right) \right\} \\
& + \sup_{\phi \in C^\infty([0,T] \times \mathbb{T}^d)} \left\{ \ell_\phi(\nu) - \frac{1}{2} \int_0^T \langle \rho_s, |\nabla \phi(s)|^2 \rangle ds \right. \\
& \qquad \left. - \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left(\phi(s,y) - \phi(s,x) \right)^2 V(x) \rho_s(dy) \rho_s(dx) ds \right\} \\
& := I_0(\nu_0) + I_{dyn}(\nu).
\end{aligned} \tag{3.1}
$$

- $$\ell_\phi(\nu) = \langle \nu_T, \phi(T)\rangle - \langle \nu_0, \phi(0)\rangle - \int_0^T \left(\langle \nu_s, \partial_s \phi(s) + \frac{1}{2}\triangle \phi(s)\rangle \right) ds$$
$$- \int_0^T \left(\langle \rho_s, V\rangle \langle \nu_s, \phi(s)\rangle + \langle \rho_s, \phi(s)\rangle \langle \nu_s, V\rangle - \langle \nu_s, \phi(s) V\rangle \right) ds. \tag{3.2}$$

- That is, for any closed set $F \subset D([0,T], \mathbb{H}^{-(5+3D)})$,

$$\limsup_{N\to\infty} \frac{N}{a^2(N)} \log P_\gamma^N(\widetilde{\eta}^N \in F) \le -\inf_{\nu\in F} I(\nu) \tag{3.3}$$

and for any open set $O \subset D([0,T], \mathbb{H}^{-(5+3D)})$,

$$\liminf_{N\to\infty} \frac{N}{a^2(N)} \log P_\gamma^N(\widetilde{\eta}^N \in O) \ge -\inf_{\nu\in O} I(\nu). \tag{3.4}$$

- For any $\phi \in C^{1,2}([0,T] \times \mathbb{T}^d)$,

$$\langle \widetilde{\eta}_t^N, \phi(t) \rangle = \frac{1}{a(N)} \sum_{i=1}^{N} (\phi(t, x_i(t)) - \langle \rho_t, \phi(t) \rangle).$$

- We consider the exponential martingale $Z_t^{\phi,N}$ associated with $\frac{a^2(N)}{N} \langle \widetilde{\eta}_t^N, \phi(t) \rangle$. Under the condition **(A0)** holds, P_γ^N-martingale $Z_t^{\phi,N}$ has the following approximation:

$$Z_T^{\phi,N} = \exp \left\{ \frac{a^2(N)}{N} \ell_\phi(\widetilde{\eta}^N) - \frac{a^2(N)}{2N} \int_0^T \Big(\langle \mu_s^N, |\nabla \phi(s)|^2 \rangle + \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} (\phi(s,y) - \phi(s,x))^2 \, V(x) \mu_s^N(dy) \mu_s^N(dx) \Big) ds + \frac{a^2(N)}{N} \int_0^T \langle \widetilde{\eta}_s^N, \phi(s) \rangle \langle \mu_s^N - \rho_s, V \rangle ds + o\left(\frac{a^2(N)}{N} \right) \right\}. \tag{3.5}$$

- Define

$$
\begin{aligned}
\Lambda_0(\psi) =& \frac{1}{2}\left(\int_{\mathbb{T}^d} |\psi(x)|^2\gamma(x)dx - \left(\int_{\mathbb{T}^d} \psi(x)\gamma(x)dx\right)^2\right), \\
\Lambda_{dyn}(\phi) =& \frac{1}{2}\int_0^T \langle \rho_s, |\nabla\phi(s)|^2\rangle ds \\
& + \frac{1}{2}\int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} (\phi(s,y) - \phi(s,x))^2 \, V(x)\rho_s(dy)\rho_s(dx)ds, \\
\Lambda(\psi,\phi) =& \Lambda_0(\psi) + \Lambda_{dyn}(\phi)
\end{aligned}
$$

- If $\int_0^T \langle \widetilde{\eta}_s^N, \phi(s)\rangle \langle \mu_s^N - \rho_s, V\rangle ds \to 0$ in MDP sense, then

$$Z_T^{\phi,N} = \exp\left\{ \frac{a^2(N)}{N} \left(\ell_\phi(\widetilde{\eta}^N) - \Lambda_{dyn}(\phi) \right) + o\left(\frac{a^2(N)}{N} \right) \right\}.$$

- For any $\nu \in D([0,T], \mathbb{H}^{-(5+3D)})$ and the ball $B(\nu, \varepsilon)$, when $N \to \infty$, $\varepsilon \to 0$,

$$\frac{N}{a^2(N)} \log P_\gamma^N \left(B(\nu, \varepsilon) \right) \sim -I(\nu)$$

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Thank you!